IL NUOVO CIMENTO VOL. LXIV A, N. 2 21 Novembre 1969

Further Exact Results in Two-Dimensional Spinor Electrodynamics with Nonvanishing Fermion Mass (*).

P. L. F. HABERLER

CERN - Geneva

(ricevuto il 2 Luglio 1969)

Summary. — The exact axial vector current in two-dimensional electrodynamics with nonvanishing fermion mass is constructed using methods of Wilson and Brandt. It is shown that $j_\mu^{5\text{un}}(x) = \partial_\mu \varphi(x)$, where $j_\mu^{5\text{un}}(x)$ is the exact unrenormalized axial vector current and $\varphi(x)$ is a pseudoscalar canonical field. The problem of constructing the correct Lagrangian for this model is also solved.

1. – Introduction.

In a previous paper (1), which we refer to as I, an exact result for the unrenormalized current $j_\mu^{\text{un}}(x)$ in two-dimensional spinor electrodynamics was derived. In this paper the corresponding axial vector current $j_5^{\mu\text{un}}(x)$ is constructed, using again the methods of WILSON (2) and BRANDT (3). In this case, however, we have to rely more on the integral equations to specify the axial vector current. Therefore we study the axial vector photon vertex $\Pi^5_{\mu\nu}(p)$, the axial vector photon-photon vertex $F^5_{\mu\nu\lambda}(p, q)$ and the axial vector electron-electron vertex $\Gamma^5_\mu(p, p')$ in some detail. We start off with the renormalized

(*) To speed up publication, the author of this paper has agreed to not receive the proofs for correction.

(1) P. L. F. HABERLER: *Nuovo Cimento*, **63** A, 675 (1969).

(2) K. WILSON: to be published in *Phys. Rev.*

(3) R. BRANDT: *Ann. of Phys.*, **44**, 221 (1967), where one finds all other relevant references.

equations following Brandt's procedure [4] and show that most of the counter terms, needed to make $\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)$ well defined, vanish or are superfluous. We end up with unrenormalized quantities. This is to be expected in this model, since apart from the vacuum fluctuation diagrams, everything is finite [5]. The result that we get for the axial vector current serves as a check for the results of I:

$$j_5^\mu(x) = \varepsilon^{\nu\mu} j_\nu(x) . \tag{1.1}$$

It is well known [6] that in two-dimensional space-time (1.1) holds for the free currents. This is because of the simple properties of the γ-matrices [7].

It is supposed that (1.1) holds for the interacting [8] case also. In this model (1.1) is nontrivial because the currents depend explicitly on the electromagnetic field. Therefore one has to convince oneself that (1.1) is really true for the interacting case also.

Whereas in I, general principles like gauge invariance, charge-conjugation invariance and the assumption of canonical commutation relations specified $j_\mu^{\text{un}}(x)$ completely, gauge invariance, for example, determines $j_5^{\mu\text{un}}(x)$ only partly. The rest is supplied by dimensional arguments and the explict use of integral equations. Using this machinery, we are able to prove (1.1). From this and

$$\partial_\mu j^{\mu\text{un}}(x) = \varepsilon_{\nu\mu}\partial^\nu j_5^{\mu\text{un}}(x) = 0$$

and the main result of I, it follows that

$$j_5^{\mu\text{un}}(x) = \partial^\mu \varphi(x) , \tag{1.2}$$

where $\varphi(x)$ is a pseudoscalar *canonical* field and $j_5^{\mu\text{un}}(x)$ and $j^{\mu\text{un}}(x)$ are exact

---

(4) R. Brandt: UMD-910, to be published in *Phys. Rev.*

(5) This is also due to the fact that an indefinite metric is used.

(6) A. S. Wightman: in *Cargèse Lectures in Theoretical Physics, 1964*, vol. 2, edited by M. Lévy (New York, 1967); See also B. Klaiber: *Lectures on soluble models in field theory*, Göteborg preprint (1969).

(7) We are using the following definitions: $g^{00} = -g^{11} = 1$,

$$\gamma\cdot p = \gamma_\mu p^\mu , \qquad -\varepsilon_{\mu\nu} = \varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^5 = \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^0\gamma^1 .$$

(8) H. Lehmann and K. Pohlmeyer: Hamburg preprint (1969).

quantities. A similar result was also obtained by POHLMEYER and LEHMANN ([8]) for ps (pv) coupling in two-dimensional space-time, except that they did not show that $\varphi(x)$ is a canonical field.

With all these results in mind, we are able to construct the Lagrangian, which reproduces the correct field equations with the correct gauge-invariant current $j_\mu^{\text{un}}(x)$. This problem was first emphasized by SOMMERFIELD ([9]) and later by THIRRING ([10]) and his collaborators. They observed that one has to know the solution of the model in advance, to construct the corresponding Lagrangian. SOMMERFIELD ([9]) also conjectured that in the case of nonvanishing fermion mass, the task of finding the right Lagrangian should be not more difficult than for the case of vanishing fermion mass. This conjecture was based on the observation that the free-field equal-time commutators are mass independent. In I, and in this paper, we show that the results for the free-field case are true in general. But also in this model we are faced with the paradox ([11]) that we have to know the equal-time commutators beforehand. Only then are we able to construct the Lagrangian. The only difference compared to the approach of Sommerfield is the use of spacelike ξ in a quantity like $\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x+\xi)$, which simplifies the discussion considerably ([12]), but also leaves us with a noncovariant Lagrangian. This brings us to another important point, which is the rest mass of the photon. We construct the Lagrangian both for $\mu_0 = 0$ as well as $\mu_0 \neq 0$. In both cases we also remove the restriction of the indefinite metric. For the pure electromagnetic case this is conveniently done by working in the radiation gauge.

For the case of the massive vector boson, we present some very convincing arguments that we have found the right Lagrangian, but we cannot prove it. This is partly due to the fact that the proposals of WILSON and BRANDT break down for massive vector bosons ([13]).

The paper is arranged in the following way: in Sect. 2 we present a simpler derivation of the main result of I. We construct the axial vector current and derive and discuss the integral equations for $\Pi_{\mu\nu5}(p)$, $F_5^{\mu\nu\lambda}(p, q)$ and $\Gamma_5^\mu(p, p')$ in Sect. 3. In Sect. 4, the question of gauge invariance of the axial vector current is treated and some important equal-time commutation relations are calculated. In Sect. 5, the Lagrangian of the model is constructed and its consequences are displayed. We conclude with mentioning the consequences and relevance of our results for other fields of current interest. For further illustration, we give more details in the Appendices.

([9]) C. M. SOMMERFIELD: *Ann. of Phys.*, 26, 1 (1963).

([10]) F. SCHWABL, W. THIRRING and J. WESS: *Ann. of Phys.*, 44, 200 (1967).

([11]) This point will be discussed in some detail in Sect. 5.

([12]) C. R. HAGEN: *Nuovo Cimento*, 51 B, 169 (1967); 51A, 1033 (1967).

([13]) This was already observed by WILSON and BRANDT.

2. – Simpler derivation of the results of I.

Our starting point in I was the following expression for the renormalized current $j_\mu(x)$:

$$(2.1)\quad \begin{cases} j_\mu(x) = \lim\limits_{\xi\to 0} j_\mu(x;\xi)\,, \\ j_\mu(x;\xi) = \lim\limits_{\xi\to 0}\{T\bar\psi(x)\gamma_\mu\psi(x+\xi) - C_{1\mu}(\xi) - C_{2\mu\nu}(\xi)A^\nu(x) - \\ \qquad - C_{3\mu\nu\varrho}(\xi)A^{\nu,\varrho}(x) - C_{4\mu\nu\varrho\lambda}(\xi)A^{\nu,\varrho\lambda}(x) - C_{6ij\mu}(\xi):\bar\psi_i\psi_j:\}\,. \end{cases}$$

In this definition of the current, the renormalized coupling constant e was divided out for later convenience. T means the time-ordered product, because we consider at the beginning timelike as well as spacelike ξ.

Using dimensional arguments, it was found in I that $C_{4\mu\nu\varrho\lambda}(\xi)$ behaves like ξ for $\xi\to 0$. As is well known, $C_{4\mu\nu\varrho\lambda}$ is directly related to the photon renormalization constant Z_3. If one calculates $C_{4\mu\nu\varrho\lambda}(\xi)$ to lowest order, one finds that it does not vanish, as one would expect. This is due to the fact that $C_{4\mu\nu\varrho\lambda}(\xi)$ multiplies a function which behaves like ξ^{-1} for $\xi\to 0$. Nevertheless $C_{4\mu\nu\varrho\lambda}(\xi)$ seems to be superfluous at least for three reasons. Firstly, calculating the fourth-order contribution to $C_{4\mu\nu\varrho\lambda}(\xi)$ gives a vanishing result. Therefore there is some indication that Z_3 is given exactly by ([14])

$$Z_3 = \frac{1}{1+e_0^2/6\pi m_0^2}\,,$$

i.e. Z_3 receives the only contribution from the lowest-order diagram, which is the simple bubble.

Secondly, $C_{4\mu\nu\varrho\lambda}$ is only motivated because one wants to have

$$(2.2)\qquad \frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial p_\beta}\Pi^{\mu\nu}(p)|_{p=0} = 0$$

in the four-dimensional case, because it ensures that the residue of the pole of the photon propagator is one and not a logarithmically divergent quantity ([15]). In two-dimensional space-time, condition (2.2) is not any more necessary because Z_3 is finite there. Thus there is no need of renormalization.

The third reason to drop $C_{4\mu\nu\varrho\lambda}$ in the definition of the current is the most important one. It introduces a wrong behaviour of $\Pi_{\mu\nu}(p)$ when letting $p\to\infty$.

([14]) P. L. F. Haberler and I. Saavedra: *Nuovo Cimento*, **49** A, 194 (1967).

([15]) See, however, the approach of K. Johnson, R. Willey and M. Baker: *Phys. Rev.*, **163**, 1699 (1967), who present some arguments that Z_3 is finite.

In fact $\Pi_{\mu\nu}(p)$ blows up in this case, which clearly contradicts the fact that the so-called Schwinger term is finite in this model. From this, we see that renormalization can introduce a nonphysical effect, thus supporting ref. ([15]).

Because of all these facts, we remove $C_{\varepsilon\mu\nu\varrho\lambda}(\xi)$ from the definition of the current. Now, using the results of I (eq. (3.34)), one finds the following expression for $j_\mu(x)$ ([16]):

$$(2.3)\qquad j_\mu(x;\xi)=\tfrac{1}{2}[\bar\psi(x)\gamma_\mu\psi(x+\xi)-\gamma_\mu\psi(x)\bar\psi(x+\xi)]-\\ -eJ_\mu(\xi)(\xi\cdot A)+\big(1-Z_1^{-1}(\xi)\big)j_\mu(x)\,,$$

using

$$\lim_{\xi\to 0} j_\mu(x;\xi)=j_\mu(x)$$

gives

$$(2.4)\qquad j_\mu(x;\xi)=Z_1(\xi)\tfrac{1}{2}[\bar\psi(x)\gamma_\mu\psi(x+\xi)-\gamma_\mu\psi(x)\bar\psi(x+\xi)]-eZ_1(\xi)\,J_\mu(\xi)(\xi\cdot A)\,.$$

Now we transform back to the unrenormalized quantities which we denote by the superscript « un ». We have

$$(2.5)\qquad \begin{cases} A_\mu=A_\mu^{\text{un}}Z_3^{-\frac{1}{2}},\\ e=Z_3^{\frac{1}{2}}e_0\,,\\ \psi=Z_1^{-\frac{1}{2}}\psi^{\text{un}},\\ \Box A_\mu^{\text{un}}=Z_3^{\frac{1}{2}}ej_\mu^{\text{un}}=e_0 j_\mu^{\text{un}}\,. \end{cases}$$

Therefore we find

$$(2.6)\qquad j_\mu^{\text{un}}(x;\xi)=\tfrac{1}{2}[\bar\psi^{\text{un}}(x)\gamma_\mu\psi^{\text{un}}(x+\xi)-\gamma_\mu\psi^{\text{un}}(x)\bar\psi^{\text{un}}(x+\xi)]-e_0Z_1(\xi)J_\mu(\xi)(\xi,A)\,.$$

Now we take ξ spacelike, because we do not want to destroy the canonical for-

([16]) $Z_1(\xi)$ is also completely finite in perturbation theory, but one has to keep μ_0, the rest mass of the photon, finite. In the limit $\mu_0\to 0$, $Z_1(\xi)$ diverges. But there exists a gauge where the zero-order propagator is given by $D_{\mu\nu}^{(0)}=(g_{\mu\nu}-2k_\mu k_\nu/k^2)/\mu_0^2-k^2$. In this case $Z_1(\xi)|_{\mu_0=0}$ is finite. Since $Z_1(\xi)$ drops out of (2.9), these ambiguities do not bother us. But to have everything well defined it is understood that the limit $\mu_0\to 0$ is taken at the very end. Quantum electrodynamics as limit of a vector-meson theory is discussed in W. THIRRING: *Principles of Quantum Electrodynamics* (New York, 1958); For a recent work on the subject, see also C. DE CALAN, R. STORA and W. ZIMMERMANN: *Lett. Nuovo Cimento*, 1, 877 (1969). As a final remark, we want to stress that electrodynamics can be formulated in a fully consistent way if one works in the radiation gauge. See the remarks of Sect. 5 and the remarks of GRIFFITHS and BENDER: Harvard preprint (1969).

malism and use the following formula of I ([17]):

$$\lim_{\xi_1 \to 0} \xi_1 J_1(\xi_1) = -\frac{Z_1^{-1}}{\pi} \tag{2.7}$$

and obtain

$$j_\mu^{\text{un}}(x) = \frac{1}{2} \lim_{\xi \to 0} [\bar{\psi}^{\text{un}}(x)\gamma_\mu \psi^{\text{un}}(x+\xi) - \gamma_\mu \psi^{\text{un}}(x)\bar{\psi}^{\text{un}}(x+\xi)] - \frac{e_0}{\pi} g_{1\mu} A^1(x) . \tag{2.8}$$

Equation (2.8) is a surprising result, because it agrees with the result of THIRRING *et al.* ([10]) for the case $m_0 = 0$. From (2.8) it is very easy to obtain the equal-time commutator of the currents:

$$[j_0(x), j_1(x')]_{t=t'} = \frac{i}{\pi} \partial_1^x \delta(x - x') . \tag{2.9}$$

In this case we were not forced to use eqs. (3.58)-(3.61) of I. Since $j_0(x)$ is not dependent on $A^{\text{un}}(x)$, the calculation of the equal-time limits is straightforward ([16]).

3. – The axial vector current.

In the following the axial vector current is constructed using thereby very much the work of BRANDT ([4]) on a related subject.

The basic vectors which are needed to obtain a finite axial vector current are the following ([18]):

$$\varepsilon^{\mu\alpha} A_\alpha(x) , \qquad \varepsilon^{\mu\alpha} : A_\alpha \partial_\varrho A^\varrho : (x) , \qquad \varepsilon^{\mu\alpha} \partial_\alpha \partial_\nu A^\nu(x) . \tag{3.1}$$

Correspondingly, the operator product expansion ([2,4]) of the quantity $\psi(x) \cdot \gamma_\mu \gamma_5 \psi(x+\xi)$ has the following form:

$$T\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x+\xi) \sim R_1(\xi)\varepsilon_{\mu\alpha}A^\alpha(x) + R_2(\xi)\varepsilon_{\mu\alpha} : A^\alpha \partial_\varrho A^\varrho : (x) + \\ + R_3(\xi)\varepsilon_{\mu\alpha}\partial^\alpha \partial_\nu A^\nu(x) + R_4(\xi) : \bar{\psi}(x)\gamma_\mu\gamma_5\psi(x) : , \tag{3.2}$$

([17]) A sign mistake crept in when deriving these equations in I. Understanding $\partial_1 \delta(x-y)$ as $\partial_1^y \delta(x-y)$, restores the validity of the results of I.

([18]) Since in two-dimensional space-time $\varepsilon_{\mu\nu}$ is the only antisymmetric tensor available, there are the only *physical* allowed vectors. One could also think of $R_5(\xi) \cdot \varepsilon_{\mu\alpha} : \partial^\alpha A_\varrho(x) A^\varrho(x) :$. As one easily can convince onself, $R_5(\xi)$ vanishes like ξ^2, as $\xi \to 0$, because of dimensional arguments.

where T is the time-ordered product, because we have not yet specified in which way ξ tends to zero. In the limit $\xi\to 0$, $R_i(\xi)$ become singular and reflect the singular nature of $\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)$. $:A\partial A:$ and $:\bar\psi\gamma_\mu\gamma_5\psi:$ have to be correspondingly defined.

From dimensional arguments, we have

$$\text{(3.3)}\qquad \begin{cases} R_1(\xi)\sim\xi^{-1}, \\ R_2(\xi)\sim\xi^0, \\ R_3(\xi)\sim\xi\,, \\ R_4(\xi)\sim\xi^0, \end{cases}$$

for $\xi\to 0$, within logarithmic factors.

We observe that the operator product expansion (eq. (3.2)) is ordered by dimension. It starts with the most divergent terms and so on. This ordering was suggested by WILSON ([8]) and is very useful, because in such a way one easily finds the basic vectors. Denoting the axial current by

$$\text{(3.4)}\qquad j_5^\mu(x)=:\bar\psi(x)\gamma^\mu\gamma^5\psi(x):$$

we can invert (3.2) and write

$$\text{(3.5)}\qquad j_5^\mu(x)=\lim_{\xi\to 0}\{T\bar\psi(x)\gamma^\mu\gamma^5\psi(x+\xi)+R_1'(\xi)\varepsilon^{\mu\alpha}A_\alpha(x)+ \\ +R_2'(\xi)\varepsilon^{\mu\alpha}:A_\alpha\partial_\varrho A^\varrho:(x)+R_3'(\xi)\varepsilon^{\mu\alpha}\partial_\alpha\partial_\nu A^\nu(x)+R_4'(\xi)j_5^\mu(x)\}\,.$$

The generalized Wick product $:A\partial A:$ is in itself an expression of the form (3.5) and that of $A(x)\partial\cdot A(x+\xi)$. Thus we write

$$\text{(3.6)}\qquad j_\mu^5(x)=\lim_{\xi\to 0}j_\mu^5(x;\xi)\,,$$

where

$$\text{(3.7)}\qquad j_\mu^5(x;\xi)=T\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)+E_1(\xi)\varepsilon_{\mu\alpha}A^\alpha(x)+ \\ +E_2(\xi)\varepsilon_{\mu\alpha}A^\alpha(x)\partial_\varrho A^\varrho(x+\xi)+E_3(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\partial_\nu A^\nu(x)+E_4(\xi)j_\mu^5(x)$$

with

$$E_1(\xi)=\xi^{-1},\qquad E_2(\xi)=\xi^0,\qquad E_3(\xi)\sim\xi\,,\qquad E_4(\xi)\sim\xi^0$$

for $\xi\to 0$.

This expression will be studied in full detail. For this we define the axial vector photon vertex $\Pi_{\mu\nu}^5(p)$, the axial vector photon-photon vertex $F_{\mu\nu\lambda}^5(p,q)$, and the axial vector electron-electron vertex $\Gamma^5_\mu(p, p')$ from the Fourier transforms (symbolically denoted by F) of the Green functions as follows ([4]):

$$F\langle Tj^5_\mu(x)A_\nu(y)\rangle = \Pi^5_{\mu\varrho}(p)D^\varrho_\nu(p)\,, \tag{3.8}$$

$$F\langle Tj^5_\mu(x)A_\nu(y)A_\lambda(z)\rangle = F^5_{\mu\varrho\sigma}(p, q)D^\varrho_\nu(p)D^\sigma_\lambda(q)\,, \tag{3.9}$$

$$F\langle Tj^5_\mu\psi(y)\bar\psi(z)\rangle = G(p)\Gamma^5_\mu(p, p')G(p') + \Pi^5_{\mu\varrho}(p+p')D^{\varrho\sigma}(p+p')G(p)\Gamma_\sigma(p, p')G(p')\,, \tag{3.10}$$

where $G(p)$, $D_{\mu\nu}(q)$ and $\Gamma_\sigma(p, p')$ are the fermion Green function, the photon Green function and the vertex function, respectively.

Equations (3.6) and (3.7) thus lead to the integral representations

$$\Pi^5_{\mu\nu}(p) = \int\frac{\mathrm{d}^2k}{(2\pi)^2}\{ie\,\mathrm{Tr}\,\gamma_\mu\gamma_5 G(k)\Gamma_\nu(k, k-p)G(k-p) + E_1(k)\varepsilon_{\mu\nu} - \varepsilon_{\mu\alpha}p^\alpha p_\nu E_3(k) + E_4(k)\Pi^5_{\mu\nu}(p)\}\,, \tag{3.11}$$

$$F^5_{\mu\nu\lambda}(p, q) = \int\frac{\mathrm{d}^2k}{(2\pi)^2}\Big[i\,\mathrm{Tr}\,\gamma_\mu\gamma_5 G(k)\,\Theta_{\nu\lambda}(k, p, q)G(k+p+q) + i\int\frac{\mathrm{d}^2k'}{(2\pi)^2}E_2(k+k')\varepsilon^\alpha_\mu k^{\beta'}H_{\alpha\beta\nu\lambda}(k', p, q) + E_4(k)F^5_{\mu\nu\lambda}(p, q)\Big]\,, \tag{3.12}$$

$$\Gamma^5_\mu(p, p') = \gamma_\mu\gamma_5 + \int\frac{\mathrm{d}^2k}{(2\pi)^2}\Big[i\,\mathrm{Tr}\,\gamma_\mu\gamma_5 G(k)\mathfrak{H}(k, p, p')G(k-p+p') + i\int\frac{\mathrm{d}^2k'}{(2\pi)^2}E_2(k+k')\varepsilon_{\mu\alpha}K^{\alpha\beta}(k', p, p')k'_\beta + E_4(k)\Gamma^5_\mu(p, p')\Big]\,, \tag{3.13}$$

where ([19])

$$H_{\alpha\beta\nu\lambda} = [g_{\alpha\nu}g_{\beta\lambda}\delta(k'+q) + g_{\alpha\lambda}g_{\beta\nu}\delta(k'+p)](2\pi)^2 + D(k')\Pi_{\alpha\beta\nu\lambda}(-k'-p-q, k', p, q)D(k'+p+q)\,. \tag{3.14}$$

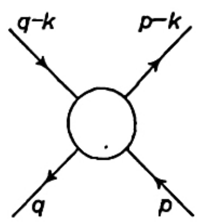

Fig. 1. – Proper electron-electron scattering amplitude.

$\Pi_{\alpha\beta\nu\lambda}$ is the proper photon-photon scattering amplitude which we discussed in Appendix B of I. $\mathfrak{H}(k, p, p')$ is the proper electron-electron scattering amplitude (see Fig. 1), $\Theta_{\nu\lambda}(k, p, q)$ is the proper electron-photon scattering amplitude $[e(k)+\gamma(p, \mu)\to e(k-p-q)+\gamma(q, \nu)]$ which satisfies the Bose symmetry condition

$$\Theta_{\mu\nu}(k, p, q) = \Theta_{\nu\mu}(k, p, q)$$

([19]) We have written $D_{\mu\nu}(k)\equiv g_{\mu\nu}D(k^2) + k_\mu k_\nu D'(k^2)$ and used the fact that $k_{1\mu}\,\pi^{\mu\nu\alpha\beta}(k_1k_2k_3k_4) = 0$ etc.

and the on-shell divergence conditions

$$p^\mu \Theta_{\mu\nu}(k, p, q) = q^\nu \Theta_{\mu\nu}(k, p, q) = 0$$

(on the electron mass shell)

$$(3.15)\qquad K_{\alpha\gamma}(k', p, p') = D_{\alpha\varrho}(p-p'-k')\Theta^{\varrho\sigma}(p', p-p'-k', k')D_{\sigma\gamma}(k') .$$

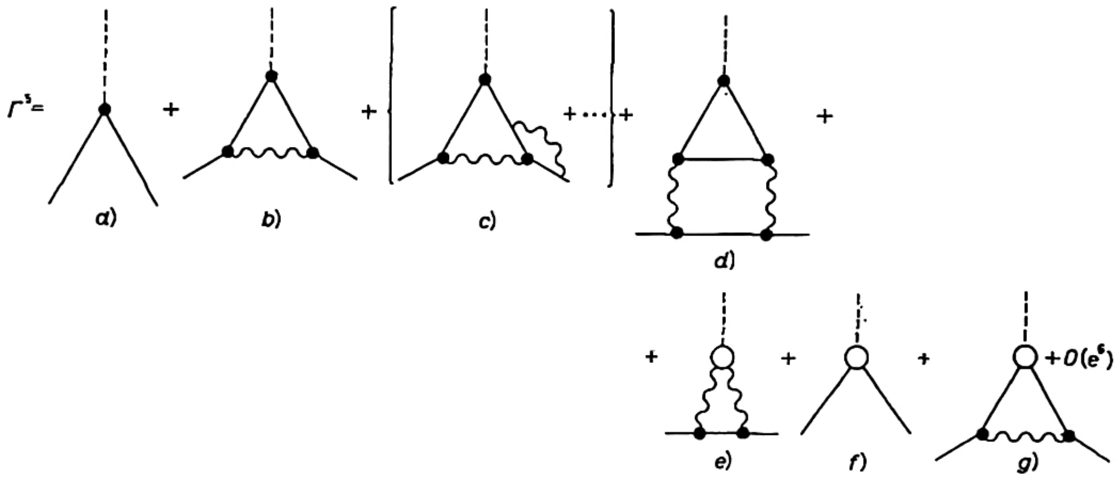

Fig. 2. – The axial vector electron-electron vertex in orders e^2 and e^4. *a)* $\gamma_\mu\gamma_5$ term; *b)*-*d)* first term of integral (3.13); *e)* second term of integral (3.13); *f)*, *g)* third term of integral (3.13). For further details, see ref. ([4]).

Equation (3.13) is graphically illustrated in Fig. 2. From eqs. (3.12) and (3.14), we already obtain an important result. Since $H_{\alpha\beta\nu\lambda}$ is multiplied by k'_β, the third term of (3.14) drops out because of

$$k'^\beta \Pi_{\alpha\beta\nu\lambda}(-k'-p-q, k', p, q) = 0$$

leaving us with the lowest-order contribution which will be explicitly calculated below.

The vertex functions in any order can be found from (3.11)-(3.14) by substituting the appropriate pure electrodynamic functions G, Γ_ν, $\Theta_{\nu k}$, $D_{\mu\nu}$, $\Pi_{\alpha\beta\gamma\delta}$, and $\mathfrak{H}$ as well as the appropriate lower-order axial vertices in the right-hand sides. The as yet unspecified subtraction functions E_i are to be chosen in each order so that the resulting integrands yield finite integrals. They are, *a priori*, otherwise arbitrary. In this context the arbitrariness in the E_i is the arbitrariness in the points at which the renormalization subtractions are made.

To choose a specific set $\{E_i\}$ one places normalization conditions in the functions (3.11)-(3.13). The number of these conditions depends on the superficial divergences ν of the functions in question. In our case $\nu_\Pi = 1$, $\nu_\Gamma = \nu_\Gamma = 0$

(*i.e.* by naïve power counting), therefore we expect only one subtraction. This is indeed true as will be seen in the following. In four-dimensional space-time $\nu_{\Pi}=3$, $\nu_{F}=2$, $\nu_{\Gamma}=1$. Therefore the following conditions are necessary for the finiteness of $\Pi^5_{\mu\nu}$, $F^5_{\mu\nu\lambda}$, Γ^5_{μ}:

$$\Pi^5_{\mu\nu}(0)=0\,, \tag{3.16}$$

$$\partial_\alpha \Pi^5_{\mu\nu}(p)|_{p=0}=0\,, \tag{3.17}$$

$$\partial_\alpha\partial_\beta \Pi^5_{\mu\nu}(p)|_{p=0}=0\,, \tag{3.18}$$

$$F^5_{\mu\nu\lambda}(0,0)=0\,, \tag{3.19}$$

$$\frac{\partial}{\partial p^\alpha} F^5_{\mu\nu\lambda}(p,0)|_{p=0}=\frac{\partial}{\partial q^\alpha} F^5_{\mu\nu\lambda}(0,q)|_{q=0}=0\,, \tag{3.20}$$

$$\Gamma^5_\mu(p,p')\Big|_{\substack{\gamma\cdot p=\gamma\cdot p'=m\\ p=p'}}=\gamma_\mu\gamma_5\,. \tag{3.21}$$

In two dimensions only (3.10) is required. It guarantees the gauge invariance of $\Pi^5_{\mu\nu}(p)$, which does not vanish, unlike in the four-dimensional case. This peculiarity of two-dimensional space-time results from

$$\Pi^{5\text{un}}_{\mu\nu}(p)=\Pi^{\text{un}}_{\nu\varrho}(p)\,\varepsilon_\mu \tag{3.22}$$

as we shall see below.

The other conditions should hold without subtractions, in other words, some of the E_i should vanish. Therefore let us impose (3.16)-(3.21) on our integral equations (3.11)-(3.13).

We start with (3.11) and get from (3.16)

$$E_1(k)\varepsilon_{\mu\nu}=-ie\,\mathrm{Tr}\,\gamma_\mu\gamma_5 G(k)\Gamma_\nu(k,k)G(k)=ie^2\,\mathrm{Tr}\,\gamma_\mu k_5\frac{\partial}{\partial k_\nu}G(k)\,. \tag{3.23}$$

Equation (3.17) gives rise to

$$\frac{\partial}{\partial p_\alpha} ie\,\mathrm{Tr}\,\gamma_\mu\gamma_5 G(k)\Gamma_\nu(k,k-p)G(k-p)|_{p=0}=0 \tag{3.24}$$

and (3.18)

$$\frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial p_\beta} ie\,\mathrm{Tr}\,\gamma_\mu\gamma_5 G(k)\Gamma_\nu(k,k-p)G(k-p)|_{p=0}=E_3(k)(\varepsilon_{\mu\alpha}g_{\nu\beta}+\varepsilon_{\mu\beta}g_{\alpha\nu})\,. \tag{3.25}$$

In Appendices A and B we discuss these equations in the two lowest orders [20].

---

[20] P. L. F. HABERLER: *Acta Phys. Austr.*, 25, 350 (1967). We would like to remark that our definition of the vacuum polarization tensor $\Pi_{\mu\nu}(p)$ differs in sign from that in J. D. BJORKEN and S. D. DRELL: *Relativistic Quantum Fields* (New York, 1965). Accordingly the photon propagator is defined in our case by $D_{\mu\nu}(k)=D_{F\mu\nu}(k)+D_{F\mu\lambda}(k)\,\Pi^{\lambda\sigma}(k)\,D_{\sigma\nu}(k)$ whilst the above-mentioned authors use $D^{\text{B.D.}}_{\mu\nu}(k)=D_{F\mu\nu}(k)-D_{F\mu\lambda}(k)\,\Pi^{\text{B.D.}}_{\lambda\sigma}(k)\,D^{\text{B.D.}}_{\sigma\nu}(k)$.

From this we see that the same remarks for E_3 are in order as for $C_{4\mu\nu\varrho\lambda}$ in Sect. 2. Since we do not want to run into contradiction with the limit $p\to\infty$ for $\Pi^5_{\mu\nu}(p)$ (or, in other words, with the limit $m_0\to 0$), we drop $E_3(\xi)$ as well as condition (3.18). Therefore we already got rid of one subtraction constant:

$$E_3(\xi) \tag{3.26}$$

is withdrawn from (3.7).

We now turn to conditions (3.19) and (3.20) and impose them on (3.12). We obtain

$$\int \mathrm{d}^2k\,\operatorname{Tr}\gamma_\mu\gamma_5 G(k)\,\Theta_{\nu\lambda}(k,p,q)\,G(k+p+q)=0\,, \tag{3.27}$$

$$\left\{\begin{aligned}&\frac{\partial}{\partial p^\varrho}\,i\,\operatorname{Tr}\gamma_\mu\gamma_5 G(k)\,\Theta_{\nu\lambda}(k,p,0)\,G(k+p)|_{p=0}=i\varepsilon_{\mu\lambda}g_{\varrho\nu}E_2(k)\,,\\&\frac{\partial}{\partial q^\varrho}\,i\,\operatorname{Tr}\gamma_\mu\gamma_5 G(k)\,\Theta_{\nu\lambda}(k,0,q)\,G(k+q)|_{q=0}=i\varepsilon_{\mu\nu}g_{\varrho\lambda}E_2(k)\,.\end{aligned}\right. \tag{3.28}$$

Here we have made use of the fact that

$$H_{\alpha\beta\nu\lambda}=(2\pi)^2[g_{\alpha\nu}g_{\beta\lambda}\delta(k'+q)+g_{\alpha\lambda}g_{\beta\mu}\delta(k'+p)]\,. \tag{3.29}$$

In Appendix C the lowest-order contributions to (3.27) and (3.28) is calculated. From this we have the result that

$$E_2^{(2)}(\xi)=0\,. \tag{3.30}$$

In the following Section it will be shown that it is true in general.

To get an expression for $E_4(\xi)$, we discuss eq. (3.13) in conjunction with condition (3.21). In anticipation of our result of Sect. 4, which is strongly suggested by (3.30), we put $E_2(\xi)$ equal to zero. Then we obtain

$$i\operatorname{Tr}\gamma_\mu\gamma_5 G(k)\,\mathfrak{H}(k,p,p')\,G(k-p+p')=-E_4(k)\gamma_\mu\gamma_5\,. \tag{3.31}$$

Now it is well known that [4]

$$-E_4(\xi)|_{\xi=0}=-\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,E_4(k)=Z_1^{-1}-1\,, \tag{3.32}$$

where Z_1^{-1} is the usual vertex renormalization constant, which is 1 in order e^0 and finite [16] in order e^2 and higher. In Appendix D, eq. (3.32) is explicitly checked in lowest order.

Thus we have obtained expressions for all E_i. $E_3(\xi)$ and $E_4(\xi)$ are determined completely, whilst $E_1(\xi)$ and $E_2(\xi)$ will be determined to all orders in the following Section. The E_i guarantee that the integral equations (3.11)-(3.14) are well defined and satisfy the appropriate boundary conditions.

Back in position space, we have the result that, with $E_i(\xi)$ given by the Fourier transforms of (3.23), (3.26), (3.30) and (3.31), the limit (3.6) exists and yields a finite local axial vector field operator. In particular, all of the renormalized Green functions

$$\langle Tj^5_\mu(x)\psi(x_1)\ldots\psi(x_n)\bar{\psi}(y_1)\ldots\bar{\psi}(y_n)A_{\nu_1}(z_1)\ldots A_{\nu_m}(z_m)\rangle \tag{3.33}$$

are finite in each order or perturbation theory. Following the discussion of ref. [4], we find that (3.7) is unique to within overall constant factor.

4. – Gauge invariance of the axial vector current.

In the last Sections, the unique local axial vector current (3.6) was constructed, essentially using only the requirements of finiteness and Lorentz covariance. Using (3.7), (3.26) and (3.32), we get the following expression for $j^5_\mu(x;\xi)$:

$$\begin{aligned} j^5_\mu(x;\xi) = T\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x+\xi) + E_1(\xi)\varepsilon_{\alpha\mu}A^\alpha(x) + \\ + E_2(\xi)\varepsilon_{\mu\alpha}A^\alpha(x)\partial_\varrho A^\varrho(x+\xi) + (1-Z_1)j^5_\mu(x)\,. \end{aligned} \tag{4.1}$$

In the following, the condition of local gauge invariance, *i.e.* the invariance under

$$\left\{\begin{aligned} &\psi(x) \;\;\to \exp[-ie\Lambda(x)]\,\psi(x)\,, \\ &A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)\,, \end{aligned}\right. \tag{4.2}$$

will be imposed on $j_\mu(x;\xi)$. It will be shown that $j^5_\mu(x;\xi)$ is indeed gauge invariant and, conversely, that the requirements of gauge invariance are sufficient to determine E_1 and E_2. In the course of the analysis, we construct explicit expressions for $E_1(\xi)$ and $E_2(\xi)$.

The mild assumption is made that the transformation (4.2) can be taken inside the $\xi\to 0$ limit in (4.1).

Let us now determine the consequences of gauge invariance. Under (4.2) the first term in (4.1) requires the factor

$$\begin{aligned} \exp\left[ie[\Lambda(x)-\Lambda(x+\xi)]\right] = \\ = 1 - ie\left[\xi\cdot\partial\Lambda(x) + \frac{1}{2}(\xi\cdot\partial)^2\Lambda(x) + \frac{1}{6}(\xi\cdot\partial)^3\Lambda(x)\right] - \\ - \frac{e^2}{2}\left[(\xi\cdot\partial)\Lambda(x)(\xi\cdot\partial)\Lambda(x) + (\xi\cdot\partial)\Lambda(x)(\xi\cdot\partial)^2\Lambda(x)\right]. \end{aligned} \tag{4.3}$$

Actually we could drop all terms of $O(\xi^2)$ and higher, because we know that

$$\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x+\xi)\sim\xi^{-1} \tag{4.4}$$

for $\xi\to 0$. But we want to show that $E_2(\xi)$ vanishes like ξ^2 for ξ going to zero. The second term of (4.1) goes into itself and

$$E_1(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\Lambda(x)\,. \tag{4.5}$$

Apart from the term which goes into itself under (4.2), the third term gives rise to the following extra contributions:

$$\begin{aligned}E_2(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\Lambda(x)\partial_\varrho A^\varrho(x+\xi)+E_2(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\Lambda(x)\partial_\varrho\partial^\varrho\Lambda(x+\xi)=\\ =E_2(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\Lambda(x)\partial_\varrho A^\varrho(x)+E_2(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\Lambda(x)\partial_\varrho\partial^\varrho\Lambda(x)\end{aligned} \tag{4.6}$$

since $\xi E_2(\xi)=0$ from (3.7).

Notice that we did not drop the second term of (4.5), which would vanish because of $\partial_\varrho\partial^\varrho\Lambda(x)=0$. Thus we find that, apart from terms which vanish for $\xi\to 0$, the transformation (4.2) induces in $j^5_\mu(x;\xi)$ the change

$$\begin{aligned}\delta j^5_\mu(x;\xi)=\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x+\xi)\left[-ie\{\cdot/.\}-\frac{e^2}{2}\{\cdot\cdot/..\}\right]+E_1(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\Lambda(x)+\\ +E_2(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\Lambda(x)\partial_\varrho A^\varrho(x)+E_2(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\Lambda(x)\partial_\varrho\partial^\varrho\Lambda(x)\,.\end{aligned} \tag{4.7}$$

where $\cdot/.$ and $\cdot\cdot/..$ are short-hand notations for

$$[\xi\cdot\partial\Lambda(x)+\tfrac{1}{2}(\xi\cdot\partial)^2\Lambda(x)+\tfrac{1}{6}(\xi\cdot\partial)^3\Lambda(x)]$$

and

$$[\xi\cdot\partial\Lambda(x)(\xi\cdot\partial)\Lambda(x)+(\xi\cdot\partial)\Lambda(x)(\xi\cdot\partial)^2\Lambda(x)]\,,$$

respectively.

Our task now is to determine the restrictions imposed on E_1 and E_2 by the requirement that

$$\lim_{\xi\to 0}\delta j^5_\mu(x;\xi)=0\,. \tag{4.8}$$

We shall first determine the forms the E_i must have in order that simply $\langle 0|\delta j^5_\mu(x;\xi)|0\rangle=0$ under (4.6). Since $\langle 0|\partial_\varrho A^\varrho(x)|0\rangle=0$, we obtain

$$\begin{aligned}\langle 0|\delta j^5_\mu(x,\xi)|0\rangle=\langle 0|\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x+\xi)|0\rangle\left[-ie\{\cdot/.\}-\frac{e^2}{2}\{\cdot\cdot/..\}\right]+\\ +E_1(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\Lambda(x)+E_2(\xi)\varepsilon_{\mu\alpha}\partial^\alpha\Lambda(x)\partial_\varrho\partial^\varrho\Lambda(x)\,.\end{aligned} \tag{4.9}$$

Since (4.8) holds for arbitrary $\Lambda(x)$, by requiring $\langle 0|\delta j_\mu^5(x;\xi)|0\rangle = 0$ we obtain the conditions that the coefficients of each of $\partial_\nu^\mu \Lambda(x)$, etc., must contribute to zero to (4.9). Therefore we obtain ([21])

$$E_1(\xi)\varepsilon_{\mu\alpha} = ie\langle 0|\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)|0\rangle\,\xi_\alpha\,, \tag{4.10}$$

$$E_2(\xi)\varepsilon_{\mu\alpha} = \frac{e^2}{2}\langle 0|\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)|0\rangle\,\xi_\alpha\xi^2/2\,. \tag{4.11}$$

From (4.4) and (4.11), it follows immediately that

$$E_2(\xi) = 0\,, \tag{4.12}$$

thus generalizing the result of the last Section, where we showed that it was true in lowest order. Equation (4.10) is rewritten in the following form:

$$E_1(\xi)\varepsilon_{\mu\alpha} = -e\xi_\alpha J_{\mu5}(\xi)\,, \tag{4.10'}$$

where

$$J_{\mu5}(\xi) = -i\langle 0|\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)|0\rangle = \operatorname{Tr}\gamma_\mu\gamma_5 G(\xi)\,, \tag{4.13}$$

$G(\xi)$ being the fermion Green function. Thus ([22]) $\langle 0|\delta j_\mu^5(x;\xi)|0\rangle = 0$ implies (4.10) and (4.11). Conversely, if the E_i have the form (4.10)-(4.11), then $\langle 0|\delta j_\mu^5(x;\xi)|0\rangle = 0$. Since the forms (4.10) and (4.11) are identical with ([22]) (3.23) and (3.30), it follows simply that $\langle 0|\delta j_\mu^5(x;\xi)|0\rangle = 0$ is equivalent to the conditions (3.16)-(3.21). Together with our previous results, this shows that the gauge invariance of the vacuum expectation value of j_μ^5 is equivalent to the gauge invariance of the theory. Using the well-known arguments ([22]), one easily shows that $\langle 0|\delta j_\mu^5(x;\xi)|0\rangle$ is equivalent to

$$\delta j_\mu^5(x,\xi) = 0\,. \tag{4.14}$$

From (4.14), or already from (4.10), we deduce that

$$\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)\xi_\alpha \sim ie\xi_\alpha J_\mu^5(\xi)\,. \tag{4.15}$$

Finally, we get the following expression for $j_\mu^5(x;\xi)$ using (4.12) and (4.10'):

$$j_\mu^5(x;\xi) = T\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi) - e\xi_\alpha J_\mu^5(\xi)A^\alpha(x) + (1-Z_1)j_\mu^5(x)\,. \tag{4.16}$$

([21]) The Lorentz covariant limit $\lim_{\xi\to 0}(\xi_\alpha\xi_\beta) = \frac{1}{2}\xi^2 g_{\alpha\beta}$ was used. This is done here only for convenience, but will not be used otherwise. The $O(\xi^2)$ terms of (4.3) are not any longer included because of (4.4).

([22]) R. BRANDT: *Ann. of Phys.*, 52, 122 (1969), and UMD-673.

From (4.16) we see that $-e\xi_\alpha J^5_\mu(\xi)A_\alpha(x)$ remains to be calculated. To do this, we restrict ourselves to spacelike ξ:

$$\xi^\mu = (0, \xi^1)\,, \tag{4.17}$$

$$-e\xi\cdot AJ_{\mu5}(\xi) = -e\xi_1 A_1 Z_1 \operatorname{Tr}\gamma_\mu\gamma_5\left(-\frac{\gamma\cdot\xi}{2\pi\xi^2}\right) = \tag{4.18}$$

$$= -\frac{e}{\pi}Z_1\xi_1A_1\xi^\varrho\varepsilon_{\mu\varrho}/\xi^2 = \frac{e}{\pi}Z_1\xi_1A_1\xi^1\varepsilon_{\mu1}/\xi_1^2 = -\frac{e}{\pi}Z_1A_1\varepsilon_{\mu1}\,,$$

where eqs. (4.13), (2.7) and ([17])

$$J^5_\mu(\xi) = -\int d\varkappa\big(\delta(\varkappa-m)+\sigma(\varkappa)\big)\operatorname{Tr}\gamma_\mu\gamma_5 S_0(\xi) = -Z_1\operatorname{Tr}\gamma_\mu\gamma_5 S_0(\xi) \tag{4.19}$$

were used.

Therefore we find

$$j^5_\mu(x;\xi) = \bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi) - \frac{e}{\pi}Z_1A_1\varepsilon_{\mu1} + (1-Z_1)\,j^5_\mu(x)\,. \tag{4.20}$$

Passing now to the unrenormalized quantities, using (2.5) and bringing $j^5_\mu(x)$ on the other side, we end up with the following expression for the unrenormalized axial vector current:

$$j^{5\mathrm{un}}_\mu(x;\xi) = \bar\psi^{\mathrm{un}}(x)\gamma_\mu\gamma_5\psi^{\mathrm{un}}(x+\xi) - \frac{e_0}{\pi}A_1^{\mathrm{un}}\varepsilon_{\mu1}\,. \tag{4.21}$$

$j^{5\mathrm{un}}_\mu(x)$ is then defined by (3.6). Following the procedure of I, we observe that

$$j^{5\mathrm{un}}_\mu(x) = \lim_{\xi\to0} j^{5\mathrm{un}}_\mu(x+\xi, -\xi) \tag{4.22}$$

also holds. Therefore we define

$$\tilde{j}^{5\mathrm{un}}_\mu(x) = \tfrac{1}{2}\lim_{\xi\to0}\,[j^{5\mathrm{un}}_\mu(x;\xi) + j^{5\mathrm{un}}_\mu(x+\xi,-\xi)]\,. \tag{4.23}$$

Finally we obtain

$$\tilde{j}^{5\mathrm{un}}_\mu(x) = \lim_{\xi\to0}\{\tfrac{1}{2}[\bar\psi^{\mathrm{un}}(x)\gamma_\mu\gamma_5\psi^{\mathrm{un}}(x+\xi) - \gamma_\mu\gamma_5\psi^{\mathrm{un}}(x)\,\bar\psi^{\mathrm{un}}(x+\xi)]\} - \frac{e_0}{\pi}A_1^{\mathrm{un}}\varepsilon_{\mu1}\,. \tag{4.24}$$

Comparing (4.24) with (2.8), we obtain our main result

$$j^{5\text{un}}_{\mu}(x) = \varepsilon_{\nu\mu}\, j^{\nu\text{un}}(x)\,. \tag{4.25}$$

From I, we know that

$$\partial_{\mu} j^{\mu\text{un}}(x) = 0$$

and therefore

$$\varepsilon_{\nu\mu}\,\partial^{\nu} j^{\mu\text{un}}_{5}(x) = 0\,, \tag{4.26}$$

i.e. the curl of the axial vector current vanishes. Therefore we obtain

$$j^{5\text{un}}_{\mu}(x) = \partial_{\mu}\varphi(x)\,, \tag{4.27}$$

where $\varphi(x)$ is a pseudoscalar field. That this is true for the free-field case was shown in ref. (6), see also ref. (8) for the interacting case.

Now we want to show that $\varphi(x)$ is a canonical field. To prove this we use (2.9) and (4.25)

$$[j^{5\text{un}}_{0}(x),\, j^{5\text{un}}_{1}(x')]_{t=t'} = [j^{\text{un}}_{1}(x),\, j^{\text{un}}_{0}(x')]_{t=t'} = -\frac{i}{\pi}\,\partial^{x'}_{1}\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{4.28}$$

On the other hand

$$[j^{5\text{un}}_{0}(x),\, j^{5\text{un}}_{1}(x')]_{t=t'} = \partial^{x'}_{1}[\partial_{0}\varphi(x),\, \varphi(x')]_{t=t'}\,. \tag{4.29}$$

Combining (4.28) and (4.29) we obtain

$$[\partial_{0}\varphi(x),\, \varphi(x')]_{t=t'} = -\frac{i}{\pi}\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{4.30}$$

i.e. $\varphi(x)$ is a canonical field.

The interesting feature of eqs. (4.25), (4.27) and (4.30) is that they are valid for the interacting currents.

Before turning to other equal-time commutators, we notice that

$$\partial_{\mu} j^{\mu\text{un}}_{5}(x,\xi) = im_{0}[\bar{\psi}^{\text{un}}(x)\gamma_{5}\psi^{\text{un}}(x+\xi) - \gamma_{5}\psi^{\text{un}}(x)\bar{\psi}^{\text{un}}(x+\xi)] - \frac{e_{0}}{\pi}\,\varepsilon_{\mu 1}\,\partial^{\mu}A^{\text{un}}_{1}(x)\,. \tag{4.31}$$

This follows from the field equations (4.15) and from

$$A^{\mu}\varepsilon_{\mu\varrho}A^{\varrho} = 0\,.$$

For later use the following equal-time commutators are calculated:

$$[j_\mu^{5\mathrm{un}}(x),\, \psi^{\mathrm{un}}(x')]_{t=t'} = -\gamma_0\gamma_\mu\gamma_5\psi^{\mathrm{un}}(x)\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{4.32}$$

In particular

$$[j_0^{5\mathrm{un}}(x),\, \psi^{\mathrm{un}}(x')]_{t=t'} = -\gamma_5\psi^{\mathrm{un}}(x)\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{4.33}$$

Furthermore we have

$$[j_\mu^{5\mathrm{un}}(x),\, A_\nu^{\mathrm{un}}(x')]_{t=t'} = 0\,, \tag{4.34}$$

$$[j_0^{5\mathrm{un}}(x),\, j_1^{\mathrm{un}}(x')]_{t=t'} = [j_1^{5\mathrm{un}}(x),\, j_0^{\mathrm{un}}(x')]_{t=t'} = 0\,, \tag{4.35}$$

$$[j_0^{5\mathrm{un}}(x),\, j_1^{5\mathrm{un}}(x')]_{t=t'} = \frac{i}{\pi}\,\partial_1^x\delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{4.36}$$

5. – Construction of the proper Lagrangian.

In this Section, we are going to construct the Lagrangian for this model. As one knows (9,10), this is a nontrivial problem. Due to the fact that the current (23) $j_\mu(x)$ depends explicitly on the electromagnetic field, the usual Lagrangian does not give rise to the correct field equations, with the correct gauge-invariant current. Therefore, one has to alter the Lagrangian. Lorentz invariance of the Lagrangian is also broken, due to the fact that we are using spacelike ξ only.

In the following, three cases are discussed:

a) Lagrangian in the Gupta-Bleuler gauge; this necessitates the use of indefinite metric;

b) Lagrangian in the radiation gauge;

c) Lagrangian for a neutral vector meson, which carries a nonzero rest mass, in interaction with fermions.

We start with case *a*). The usual Lagrangian is given by (24)

$$\mathscr{L} = -\frac{1}{2}A_{\mu,\nu}A^{\mu,\nu} + \frac{\mu_0^2}{2}A_\mu A^\mu - e_0 A_\mu J^\mu + \frac{i}{2}\bar\psi\gamma^\mu\partial_\mu\psi - \frac{i}{2}\partial_\mu\bar\psi\gamma^\mu\psi - m_0 :\bar\psi\psi:\,, \tag{5.1}$$

(23) In the following all quantities are understood to be unrenormalized. For notational convenience we drop the superscript « un ».

(24) S. Schweber: *An Introduction to Relativistic Quantum Field Theory* (Evanston, Ill., 1961). The sign of e is reversed in our case.

where

$$:\bar{\psi}\psi:(x) = \lim_{\xi\to 0}\left[\bar{\psi}(x)\psi(x+\xi) - \langle 0|\bar{\psi}(x)\psi(x+\xi)|0\rangle\right]. \tag{5.2}$$

We have also included a mass term for the photon, representing the fact that μ_0 is put to zero at the very end of the calculations.

$J_\mu(x)$ is defined by

$$J_\mu(x) = \lim_{\xi\to 0}\left[\bar{\psi}(x+\xi)\gamma_\mu\psi(x) - \langle 0|\bar{\psi}(x+\xi)\gamma_\mu\psi(x)|0\rangle\right]. \tag{5.3}$$

To obtain the right Lagrangian, it is not enough to replace the nongauge invariant current $J_\mu(x)$ by $j_\mu(x)$, eq. (2.8), since

$$[j_1(x),\, \partial'_0 A_1(x')]_{t=t'} = -\frac{ie_0}{\pi}\,\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{5.4}$$

The correct Lagrangian density is given by (10)

$$\mathscr{L} = -\frac{1}{2}A_{\mu,\nu}A^{\mu,\nu} + \frac{\mu_0^2}{2}A_\mu A^\mu + \frac{e_0^2}{2\pi}A_1^2 - e_0 A_\mu j^\mu + \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu\psi - m_0:\bar{\psi}\psi:. \tag{5.5}$$

The term $(e_0^2/2\pi)A_1^2$ destroys the Lorentz invariance and gives also rise to a nonpositive definite Hamiltonian.

Both of these troubles can be cured if one works in the radiation gauge. With it, we turn to the discussion of case *b*).

The radiation gauge is characterized by

$$\partial^l A^l = 0\,, \qquad l = 1, 2, 3. \tag{5.6}$$

In two-dimensional space-time it follows from (5.6) that (25)

$$A^l = A^1 = 0\,. \tag{5.7}$$

Since $j_\mu(x)$ only depends on $A_1(x)$ and $A_1(x)$ is zero in this case, it follows then that nothing has to be changed in the Lagrangian:

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e_0 A_\mu j^\mu + \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu\psi - m_0:\bar{\psi}\psi:, \tag{5.8}$$

where $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$.

---

(25) See, *e.g.*, L. S. Brown: *Nuovo Cimento*, **29**, 617 (1963).

Here once more [16,26] we notice the fact that the radiation gauge plays a special role in field theory. It guarantees the positivity of the energy-momentum tensor in presence of massless particles. Also it is the only gauge where an unambiguous transition from massive to massless vector boson can be achieved.

Also we want to remark that due to the fact that the fermions have a finite rest mass, there are no troubles with the masslessness of the photon, as pointed out by ZUMINO [27] for the case of the Schwinger model.

Let us now turn to a discussion of case *c*), the interactions of massive vector bosons with massive fermions. Our main results should hold [28] in this case also. Indeed, if one studies the results of ref. [10], then one observes that the commutator $[j_0, j_1]$ does not depend on the rest mass of the vector boson. Also the vacuum polarization tensor is free from infra-red troubles, *i.e.* the limit vector-boson mass going to zero can be taken unambiguously.

We have the following Lagrangian in this case:

$$\mathscr{L} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{M_0^2}{2} B_\mu B^\mu - e_0 B_\mu J^\mu + \frac{i}{2} \bar{\psi}\gamma^\mu \partial_\mu \psi - \frac{i}{2} \partial^\mu \bar{\psi}\gamma_\mu \psi - m_0 :\bar{\psi}\psi: , \tag{5.9}$$

where

$$\begin{cases} G_{\mu,\nu} = B_{\mu,\nu} - B_{\nu,\mu} , \\ B^\mu{}_{,\mu} = 0 , \\ [B^1(x), G^{10}(x')]_{t=t'} = i\delta(\boldsymbol{x} - \boldsymbol{x}') , \\ M_0 = \text{rest mass} , \end{cases} \tag{5.10}$$

$J_\mu(x)$ is given by (5.3).

This Lagrangian gives rise to field equations with a nonconserved current. As in case *a*), we have to account for this fact, by adding a term $B_1^2(e_0^2/2\pi)$ to the Lagrangian, thereby destroying the Lorentz invariance of $\mathscr{L}$. The modified Lagrangian has the following form:

$$\mathscr{L} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{M_0^2}{2} B_\mu B^\mu + B_1^2 \frac{e_0^2}{2\pi} - e_0 B_\mu j^\mu + \frac{i}{2} \bar{\psi}\gamma^\mu \partial_\mu \psi - \frac{i}{2} \partial^\mu \bar{\psi}\gamma_\mu \psi - m_0 :\bar{\psi}\psi: . \tag{5.11}$$

---

[26] See also, J. SCHWINGER: in *Particles and Field Theory, Brandeis Lectures, 1964*, vol. 2, edited by S. DESER and K. W. FORD (Englewood Cliffs., N. J., 1965).

[27] B. ZUMINO: *Phys. Lett.*, **10**, 224 (1964).

[28] B. ZUMINO: private communication.

This gives rise to the following Hamiltonian ([10]):

$$(5.12)\qquad H=\int dx_1\mathscr{H}(x_1)=\tfrac{1}{2}\int dx_1[(G^{10})^2+(e_0J_0-G^{10}{}_{,1})^2/M_0^2+M^2B_1^2-\\ -2e_0J_1B_1+2i\bar\psi\gamma_1\partial_1\psi+m_0:\bar\psi\psi:]\,,$$

where we have introduced

$$(5.13)\qquad \begin{cases} M^2=M_0^2+\dfrac{e_0^2}{\pi}\,, \qquad j^0=J^0\,,\\ j^1=J^1-\dfrac{e_0}{\pi}B^1\,.\end{cases}$$

Equation (5.13) indicates the fact that the mass of the vector boson is changed ([29]).

Having this Hamiltonian available, we are now in the position to calculate the equations of motion of the following fields:

$$(5.14)\qquad \begin{cases} G^{10}(x)\,,\quad B^1(x)\,,\quad J^0(x)\,,\quad J^1(x)\,,\\ S(x)=:\bar\psi\psi:(x)\,,\end{cases}$$

$$(5.15)\qquad \begin{cases} \pi(x)=:\bar\psi\gamma_5\psi:(x)=\lim\limits_{\xi\to0}\{\tfrac{1}{2}[\bar\psi(x)\gamma_5\psi(x+\xi)-\gamma_5\psi(x)\bar\psi(x+\xi)]\}\,,\\ \varphi(x)\,.\end{cases}$$

From (4.27) and (4.31) we obtain for $\varphi(x)$

$$(5.16)\qquad \Box\varphi(x)=2im_0\pi(x)-\frac{e_0}{\pi}\varepsilon_{\mu1}\partial^\mu B_1(x)$$

and from (5.12), using (4.24)-(4.36) as well as (5.10), one gets

$$(5.17)\qquad \begin{cases} \dot G^{10}(x)=-M^2B^1(x)+e_0J^1(x)\,,\\ \dot B^1(x)=\left(1-\dfrac{\partial^{1^2}}{M_0^2}\right)G^{10}+\dfrac{e_0}{M_0^2}\partial^1J^0(x)\,,\\ \dot J_0(x)=-\dfrac{e_0}{\pi}\partial^1B^1(x)-\partial^1J^1(x)\,,\\ \dot J^1(x)=-\dfrac{e_0}{\pi}\dfrac{\partial^{1^2}}{M_0^2}G^{10}(x)+\dfrac{M^2}{M_0^2}\partial^1J^0(x)-2m_0i\pi(x)\,,\\ \dot\pi(x)=i[2e_0B^1S+2m_0J^1+i(\partial^1\bar\psi)\psi-i\bar\psi\partial^1\psi]\,,\\ \dot S(x)=i[2e_0B^1\pi+i(\partial^1\bar\psi)\gamma_5\psi-i\bar\psi\gamma_5\partial^1\psi]\,.\end{cases}$$

([29]) P. L. F. HABERLER: *Nuovo Cimento*, 47 A, 929 (1967).

Due to the mass term we cannot exactly diagonalize eq. (5.17). This shows that the model gives rise to a nontrivial S-matrix and one has to use other methods to solve it exactly. Unfortunately, all efforts [30] in this direction have failed.

We want to mention that eq. (5.17) might be a good starting point to construct a Sugawara [31] model. We want to return to this problem in another paper.

Another point should be made in connection with this model. Since the equal-time commutator $[j_0(x), j_1(x')]_{t=t'}$ is also given by (2.9) in this case, the following sum rule has to be true [10,28]:

$$\int \mathrm{d}a^2 \varrho^{\mathrm{un}}(a^2) = \frac{e_0^2}{\pi}, \tag{5.18}$$

where $\varrho^{\mathrm{un}}(a^2)$ is the spectral function of the vacuum polarization tensor with nonvanishing photon mass. This is the same sum rule as we have derived in I, and the same remarks are in order here [32].

6. – Conclusions.

In this work we continued the discussion of two-dimensional models describing the interaction of massive fermions with massless and massive photons, respectively. The main aim of this paper was to approach the main result of I by a different method. To do this we have studied the axial vector current *in extenso*, thereby using again the methods of Wilson and Brandt. We were able to prove the conjecture that

$$j^5_\mu(x) = \varepsilon^\nu{}_\mu\, j_\nu(x) ,$$

thereby establishing the main result of I. As a by-product, some further results were obtained. We were able to show that the $j^5_\mu(x)$ can be written in the following way:

$$j^5_\mu(x) = \partial_\mu \varphi(x) ,$$

---

(30) In this connection the following work is also relevant: B. SIMON: *Nuovo Cimento*, 59 A, 199 (1969).

(31) A. SUGAWARA: *Phys. Rev.*, **170**, 1659 (1968); C. SOMMERFIELD: to be published.

(32) This is due to the fact that the vacuum polarization tensor stays finite, letting M_0 going to zero. See in this connection ref. (10).

where $\varphi(x)$ is a pseudoscalar canonical field, thereby generalizing the free-field results [6] to the interacting case.

The construction of the proper Lagrangian was discussed in some detail. We were able to generalize the results of SOMMERFIELD [9], THIRRING *et al.* [10] and HAGEN [12] to the case of massive fermions. We also stressed the fact that the radiation gauge plays a special role in theories involving massless particles.

A short discussion of the model where massive fermions interact with massive photons showed that this model contains some nice physical features. We believe therefore that its exact solution brings us close to the answer how the dynamics of a realistic theory should look like.

* * *

I am very much indebted to Prof. V. GLASER and Dr. H. EPSTEIN for reading the manuscript and for many useful discussions.

Discussions with Dr. D. GROSS and Dr. O. NACHTMANN are gratefully acknowledged. Thanks are also due to Prof. B. ZUMINO for his kind interest in this work.

The author wants also to thank Dr. R. BRANDT for sending him his work prior to publication.

APPENDIX A

In the following the lowest-order contribution to $\Pi^5_{\mu\nu}(p)$ is calculated. From eq. (3.11) we have

$$\text{(A.1)}\qquad \Pi^{5(2)}_{\mu\nu}(p) = ie_0\int\frac{d^2k}{(2\pi)^2}\,\mathrm{Tr}\,\gamma_\mu\gamma_5\frac{1}{\gamma\cdot k-m_0}\gamma_\nu\frac{1}{\gamma\cdot k-\gamma\cdot p-m_0}-$$

$$-ie_0\int\frac{d^2k}{(2\pi)^2}\,\mathrm{Tr}\,\gamma_\mu\gamma_5\frac{1}{\gamma\cdot k-m_0}\gamma_\nu\frac{1}{\gamma\cdot k-m_0}-$$

$$-\frac{ie_0}{2}\int\frac{d^2k}{(2\pi)^2}\,\mathrm{Tr}\,\frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial p^\alpha}\gamma_\mu\gamma_5\frac{1}{\gamma\cdot k-m_0}\gamma_\nu\frac{1}{\gamma\cdot k-\gamma\cdot p-m_0}\Big|_{p=0}.$$

In (A.1) only the unrenormalized parameters appear because we work in lowest order.

The two last terms of (A.1) are the contributions from $E_1(k)$ and $E_3(k)$. It is a simple task to calculate (A.1). For the evaluation of the traces one has just to observe that

$$\text{(A.2)}\qquad \gamma_\mu\gamma_\nu = g_{\mu\nu}+\gamma_5\varepsilon_{\mu\nu}\,.$$

Using standard techniques, one finally obtains

$$\text{(A.3)}\qquad \Pi^{5(2)}_{\mu\nu}(p) = -\frac{e_0}{4\pi}\left\{-\int_0^1 \frac{\mathrm{d}z\,4z(z-1)(\varepsilon_{\mu\nu}p^2 + p_\nu \hat{p}_\mu)}{p^2 z(z-1) + m_0^2} + 2\varepsilon_{\mu\nu}\right\} + $$

$$+ \frac{e_0}{2\pi}\varepsilon_{\mu\nu} + \frac{e_0}{\pi}\frac{(\varepsilon_{\mu\nu}p^2 + p_\nu\hat{p}_\mu)}{6m_0^2} = -\frac{e_0}{\pi}(\varepsilon_{\nu\mu} - p_\nu\hat{p}_\mu/p^2)\varrho^5_{(2)}(p^2)\,,$$

where

$$\text{(A.4)}\qquad \varrho^5_{(2)}(p^2) = \int_0^1 \frac{\mathrm{d}z\,p^2 z(z-1)}{p^2 z(z-1) + m_0^2} + \frac{p^2}{6m_0^2}\,, \qquad \hat{p}_\mu = \varepsilon_{\varrho\mu}p^\varrho\,.$$

Except for the term $p^2/6m_0^2$, which arose from our normalization condition ([20])

$$\text{(A.5)}\qquad \varrho^5_{(2)}(p^2) = \varrho^{\text{un}}_{(2)}(p^2)\,,$$

where $\varrho_{(2)}(p^2)$ is defined through

$$\text{(A.6)}\qquad \Pi^{(2)}_{\mu\nu}(p) = -(g_{\mu\nu} - p_\mu p_\nu/p^2)\varrho^{\text{un}}_{(2)}(p^2)\,.$$

Furthermore we observe that

$$\text{(A.7)}\qquad p^\nu \Pi^{(2)}_{\mu\nu5}(p) = 0\,, \qquad \hat{p}^\mu \Pi^{(2)}_{\mu\nu5}(p) = 0\,.$$

This was to be expected.

Appendix B

In this Appendix the fourth-order contribution to $\Pi^{\text{un}(4)}_{\mu\nu5}(p)$ will be discussed (« un » means unrenormalized). The following is explicitly shown:

(B.1) *a*) $\Pi^{\text{un}(4)}_{\mu\nu5}(0) = 0\,,$

(B.2) *b*) $p^\nu \Pi^{\text{un}(4)}_{\mu\nu5}(p) = \hat{p}^\mu \Pi^{\text{un}}_{\mu\nu5}(p) = 0\,,$

(B.3) *c*) $\varrho^{\text{un}(4)}_{\mu\nu5}(p^2) = \varrho^{\text{un}(4)}(p^2)\,, \qquad \Pi^{\text{un}(4)}_{\mu\nu5}(p) = -\varepsilon_{\mu\varrho}\Pi^{\text{un}}_\nu \varrho^{(4)}(p)\,.$

We start with *a*). Using standard techniques ([20]), one finds

$$\text{(B.4)}\qquad \Pi^{\text{un}(4)}_{\mu\nu5}(p) = \frac{e_0^4}{(2\pi)^4}\int \mathrm{d}^2q \int \mathrm{d}^2k\,\operatorname{Tr}\gamma_\mu\gamma_5 \frac{1}{\gamma\cdot q - m_0}\gamma_\varrho \frac{1}{\gamma\cdot q + \gamma\cdot k - m_0}\cdot$$

$$\cdot\gamma^\varrho \frac{1}{\gamma\cdot q - m_0}\gamma_\nu \frac{1}{\gamma\cdot q - \gamma\cdot p - m_0}\frac{1}{k^2 - \mu_0^2} + \frac{e_0^4}{(2\pi)^4}\int \mathrm{d}^2q \int \mathrm{d}^2k\,\operatorname{Tr}\gamma_\mu\gamma_5\cdot$$

$$\cdot\frac{1}{\gamma\cdot q+\gamma\cdot p-m_0}\gamma_\nu\frac{1}{\gamma\cdot q-m_0}\gamma_\varrho\frac{1}{\gamma\cdot q+\gamma\cdot k-m_0}\gamma^\varrho\frac{1}{\gamma\cdot q-m_0}\frac{1}{k^2-\mu_0^2}+$$

$$+\frac{e_0^4}{(2\pi)^4}\int d^2q\int d^2k\,\mathrm{Tr}\,\gamma_\mu\gamma_5\frac{1}{\gamma\cdot q-m_0}\gamma^\varrho\frac{1}{\gamma\cdot k-m_0}\gamma_\nu\frac{1}{\gamma\cdot k-\gamma\cdot p-m_0}\cdot$$

$$\cdot\gamma_\varrho\frac{1}{\gamma\cdot q-\gamma\cdot p-m_0}\frac{1}{(q-k)^2-\mu_0^2}.$$

Setting $p=0$, doing the traces and taking into account the fact that the first two terms of (B.4) are equal, one finds

$$\Pi^{\mathrm{un}(4)}_{\mu\nu5}(0)=-\frac{8m_0^2e_0^4}{(2\pi)^4}\int d^2q\int d^2k\left\{\frac{(q^2+m_0^2)\varepsilon_{\mu\nu}-2q_\mu\hat{q}_\nu-2\hat{q}_\mu q_\nu}{(q^2-m_0^2)^3(k^2-m_0^2)[(k-q)^2-\mu_0^2]}+\right.$$

$$\left.+\frac{\varepsilon_{\mu\nu}k\cdot q-q_\mu\hat{k}_\nu-\hat{q}_\mu k_\nu+g_{\mu\nu}\hat{k}\cdot q}{(k^2-m_0^2)^2(q^2-m_0^2)^2[(q-k)^2-\mu_0^2]}\right\}.$$

Doing the k integration and doing some integrations by parts in q, one obtains

$$\Pi^{\mathrm{un}(4)}_{\mu\nu5}(0)=-\frac{4m_0^2e_0^4\pi^2\varepsilon_{\mu\nu}}{(2\pi)^4}\int_0^\infty dp^2\int_0^1\frac{dx}{x(x-1)}\left[-\frac{1}{(p^2+m_0^2)^2(p^2+A^2)}+\right.$$

$$\left.+\frac{2m_0^2}{(p^2+m_0^2)^3(p^2+A^2)}+\frac{1}{(p^2+A^2)(p^2+m_0^2)^2}-\frac{2m_0^2}{(p^2+m_0^2)^3(p^2+A^2)}\right]=0,$$

$$A^2\equiv[m_0^2x+\mu_0^2(1-x)]/x(1-x).$$

Next we prove (B.2); using (3.11), we find

$$\text{(B.5)}\qquad p^\nu\Pi^{\mathrm{un}(4)}_{\mu\nu5}(p)=\frac{e_0^4}{(2\pi)^4}\int d^2q\int d^2k\,\mathrm{Tr}\,\gamma_\mu\gamma_5\frac{1}{\gamma\cdot q-m_0}\gamma_\varrho\frac{1}{\gamma\cdot q+\gamma\cdot k-m_0}\cdot$$

$$\cdot\gamma^\varrho\frac{1}{\gamma\cdot q-m_0}\gamma\cdot p\frac{1}{\gamma\cdot q-\gamma\cdot p-m_0}\frac{1}{k^2-\mu_0^2}+\frac{e_0^0}{(2\pi)^4}\int d^2q\int d^2k\,\mathrm{Tr}\,\gamma_\mu\gamma_5\cdot$$

$$\cdot\frac{1}{\gamma\cdot q+\gamma\cdot p-m_0}\gamma\cdot p\frac{1}{\gamma\cdot q-m_0}\gamma_\varrho\frac{1}{\gamma\cdot q+\gamma\cdot k-m_0}\gamma^\varrho\frac{1}{\gamma\cdot q-m_0}\frac{1}{k^2-\mu_0^2}+$$

$$+\frac{e_0^4}{(2\pi)^4}\int d^2q\int d^2k\,\mathrm{Tr}\,\gamma_\mu\gamma_5\frac{1}{\gamma\cdot q-m_0}\gamma^\varrho\frac{1}{\gamma\cdot k-m_0}\cdot$$

$$\cdot\gamma\cdot p\frac{1}{\gamma\cdot k-k\cdot p-m_0}\gamma_\varrho\frac{1}{\gamma\cdot q-\gamma\cdot p-m_0}\frac{1}{(q-k)^2-\mu_0^2}.$$

Using

$$\text{(B.6)}\qquad \frac{1}{\gamma\cdot q-m_0}\gamma\cdot p\frac{1}{\gamma\cdot q-\gamma\cdot p-m_0}=\frac{1}{\gamma\cdot q-\gamma\cdot p-m_0}-\frac{1}{\gamma\cdot q-m_0},$$

i.e. the Ward identity, and making some obvious changes in the integration variables, we get

$$p^{\nu}\Pi^{\mathrm{un}(4)}_{\mu\nu 5}(p) = \frac{e_0^4}{(2\pi)^4}\int \mathrm{d}^2q\int \mathrm{d}^2k\,\frac{1}{(k-p)^2-\mu_0^2}\,\mathrm{Tr}\left\{\gamma_\mu\gamma_5\frac{1}{\gamma\cdot q-m_0}\gamma_\varrho\frac{1}{\gamma\cdot k-m_0}\cdot\right.$$

$$\cdot\gamma^\varrho\frac{1}{\gamma\cdot q-\gamma\cdot p-m_0}-\gamma_\mu\gamma_5\frac{1}{\gamma\cdot q-m_0}\gamma_\varrho\frac{1}{\gamma\cdot k-m_0}\gamma^\varrho\frac{1}{\gamma\cdot q-m_0}+$$

$$+\gamma_\mu\gamma_5\frac{1}{\gamma\cdot q-m_0}\gamma^\varrho\frac{1}{\gamma\cdot k-\gamma\cdot p-m_0}\gamma_\varrho\frac{1}{\gamma\cdot q-\gamma\cdot p-m_0}-$$

$$-\gamma_\mu\gamma_5\frac{1}{\gamma\cdot q-m_0}\gamma^\varrho\frac{1}{\gamma\cdot k-m_0}\gamma_\varrho\frac{1}{\gamma\cdot q-\gamma\cdot p-m_0}+\gamma_\mu\gamma_5\frac{1}{\gamma\cdot q-m_0}\gamma^\varrho\frac{1}{\gamma\cdot k-m_0}\cdot$$

$$\left.\cdot\gamma_\varrho\frac{1}{\gamma\cdot q-m_0}-\gamma_\mu\gamma_5\frac{1}{\gamma\cdot p+\gamma\cdot q-m_0}\gamma_\varrho\frac{1}{\gamma\cdot k-m_0}\gamma^\varrho\frac{1}{\gamma\cdot q-m_0}\right\}=0\,.$$

Unlike for the four-dimensional case, the use of (B.6) is reliable, because all integrals are convergent. In the same way one shows that $\hat{p}^{\mu}\Pi^{\mathrm{un}(4)}_{\mu\nu 5}(p)=0$, whereby, using the fact that

$$\hat{p}^{\mu}\varepsilon_{\varrho\mu}\gamma^{\varrho}=-\gamma\cdot p\,, \tag{B.7}$$

from (B.2) it follows that

$$\Pi^{\mathrm{un}(4)}_{\mu\nu 5}(p)=-(\varepsilon_{\nu\mu}-p_\nu\hat{p}_\mu/p^2)\,\varrho_5^{\mathrm{un}(4)}(p^2)\,. \tag{B.8}$$

To prove (B.3) we first give the expression for $\Pi^{\mathrm{un}(4)}_{\mu\nu}(p)$. We have

$$\Pi^{\mathrm{un}(4)}_{\mu\nu}(p)=-\frac{2e_0^4}{(2\pi)^4}\int\mathrm{d}^2q\int\mathrm{d}^2k\;\mathrm{Tr}\,\gamma_\mu\frac{1}{\gamma\cdot q-m_0}\gamma_\varrho\frac{1}{\gamma\cdot q+\gamma\cdot k-m_0}\gamma^\varrho\frac{1}{\gamma\cdot q-m_0}\cdot$$

$$\cdot\gamma_\nu\frac{1}{\gamma\cdot q-\gamma\cdot p-m_0}\,\frac{1}{k^2-\mu_0^2}-\frac{e_0^4}{(2\pi)^4}\int\mathrm{d}^2q\int\mathrm{d}^2k\;\mathrm{Tr}\,\gamma_\mu\frac{1}{\gamma\cdot q-m_0}\cdot$$

$$\cdot\gamma^\varrho\frac{1}{\gamma\cdot k-m_0}\gamma_\nu\frac{1}{\gamma\cdot k-\gamma\cdot p-m_0}\gamma_\varrho\frac{1}{\gamma\cdot q-\gamma\cdot p-m_0}\,\frac{1}{(q-k)^2-\mu_0^2}\,.$$

After having done the traces, we are left with the following expression:

$$\Pi^{\mathrm{un}(4)}_{\mu\nu}(p)=-\frac{4m_0^2e_0^4}{(2\pi)^4}\int\mathrm{d}^2q\int\mathrm{d}^2k\,\cdot \tag{B.9}$$

$$\cdot\left\{\frac{2p_\mu p_\nu-p^2g_{\mu\nu}+4k_\mu q_\nu-4p_\mu q_\nu}{[(p-q)^2-m_0^2][(k-p)^2-m_0^2](k^2-m_0^2)(q^2-m_0^2)[(k-q)^2-\mu_0^2]}+\right.$$

$$\left.+\frac{2g_{\mu\nu}(q^2+m_0^2)+4[2q_\mu(q_\nu-p_\nu)-g_{\mu\nu}q\cdot(q-p)]}{(q^2-m_0^2)^2(k^2-m_0^2)\big((k-q)^2-\mu_0^2\big)[(q-p)^2-m_0^2]}\right\}.$$

Since we know [20] that

$$\Pi^{\mathrm{un}(4)}_{\mu\nu}(p) = -(g_{\mu\nu} - p_\mu p_\nu/p^2)\varrho^{\mathrm{un}(4)}(p^2)\,, \tag{B.10}$$

it follows from (B.10) that

$$\varrho^{\mathrm{un}(4)}(p^2) = -\Pi^{\mu\,\mathrm{un}(4)}_{\mu}(p)\,. \tag{B.11}$$

From (B.11) it is straightforward to obtain

$$\begin{aligned}\varrho^{\mathrm{un}(4)}(p^2) = {}& \frac{8m_0^2e_0^4}{(2\pi)^4}\int \mathrm{d}^2q\int \mathrm{d}^2k\,\cdot \\ &\cdot\left\{\frac{4}{[(p-q)^2-m_0^2](q^2-m_0^2)(k^2-m_0^2)[(q-k)^2-\mu_0^2]} + \right.\\ &+\frac{4m_0^2}{[(p-q)^2-m_0^2](q^2-m_0^2)^2[(q-k)^2-\mu_0^2](k^2-m_0^2)} + \\ &+\frac{2m_0^2-p^2-\mu_0^2}{(k^2-m_0^2)[(k-p)^2-m_0^2](q^2-m_0^2)[(q-p)^2-m_0^2][(q-k)^2-\mu_0^2]} - \\ &\left.-\frac{1}{(k^2-m_0^2)[(k-p)^2-m_0^2](q^2-m_0^2)[(q-p)^2-m_0^2]}\right\}.\end{aligned} \tag{B.12}$$

To get $\varrho_5^{\mathrm{un}(4)}(p^2)$ we use (B.2) and (B.8) in the following way:

$$\begin{aligned}&\hat{p}^\mu p^\nu \Pi^{\mathrm{un}(4)}_{\mu\nu 5}(p) = 0\,,\\ &p^\mu \hat{p}^\nu \Pi^{\mathrm{un}(4)}_{\mu\nu 5}(p) = p^2\varrho_5^{\mathrm{un}(4)}(p^2)\end{aligned}$$

and therefore

$$\varrho_5^{\mathrm{un}(4)}(p^2) = p^\mu \hat{p}^\nu/p^2\,\Pi^{\mathrm{un}(4)}_{\mu\nu 5}(p)\,. \tag{B.13}$$

A straightforward computation gives

$$\varrho_5^{\mathrm{un}(4)}(p^2) = \varrho^{\mathrm{un}(4)}(p^2)\,.$$

Appendix C

In the following we discuss the lowest-order contribution to $F^{5(2)}_{\mu\nu\lambda}(p, q)$. From eq. (3.12) we have

$$\begin{aligned}F^{5(2)}_{\mu\nu\lambda}(p, q) = {}& i\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,\mathrm{Tr}\,\gamma_\mu\gamma_5\frac{1}{\gamma\cdot k - m_0}\,\Theta^{(2)}_{\nu\mu}(k, p, q)\,\frac{1}{\gamma\cdot k+\gamma\cdot p+\gamma\cdot q - m_0} - \\ &- i\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left[\varepsilon^\alpha_\mu q^\beta g_{\alpha\nu}g_{\beta\lambda}E_2(k-q) + \varepsilon^\alpha_\mu p^\beta g_{\alpha\lambda}g_{\beta\nu}E_2(k-p)\right],\end{aligned} \tag{C.1}$$

where

$$\Theta^{(2)}_{\nu\lambda}(k, p, q) = -e^2 \left\{ \gamma_\nu \frac{1}{\gamma \cdot k + \gamma \cdot p - m_0} \gamma_\lambda + \gamma_\lambda \frac{1}{\gamma \cdot k + \gamma \cdot q - m_0} \gamma_\nu \right\}. \tag{C.2}$$

Therefore we get

$$F^{5(2)}_{\mu\nu\lambda}(p, q) = -\frac{ie_0^2}{(2\pi)^2} \int \mathrm{d}^2k \cdot \tag{C.3}$$

$$\cdot \operatorname{Tr} \left\{ \frac{\gamma_\mu \gamma_5 (\gamma \cdot k + m_0) \gamma_\nu (\gamma \cdot k + \gamma \cdot p + m_0) \gamma_\lambda (\gamma \cdot k + \gamma \cdot p + \gamma \cdot q + m_0)}{(k^2 - m_0^2)[(k+p)^2 - m_0^2][(k+p+q)^2 - m_0^2]} + \right.$$

$$\left. + \frac{\gamma_\mu \gamma_5 (\gamma \cdot k + m_0) \gamma_\lambda (\gamma \cdot k + \gamma \cdot q + m_0) \gamma_\nu (\gamma \cdot k + \gamma \cdot p + \gamma \cdot q + m_0)}{(k^2 - m_0^2)[(k+q)^2 - m_0^2][(k+p+q)^2 - m_0^2]} \right\} -$$

$$- i \int \frac{\mathrm{d}^2k}{(2\pi)^2} [\varepsilon_{\mu\nu} q_\lambda E_2(k-q) + \varepsilon_{\mu\lambda} p_\nu E_2(k-p)].$$

Now it is a trivial matter to calculate $F^{5(2)}_{\mu\nu\lambda}(0, 0)$. Setting $p = q = 0$, one readily obtains the result that

$$F^{5(2)}_{\mu\nu\lambda}(0, 0) = 0. \tag{C.4}$$

So condition (3.19) is fulfilled. We want to remark that the first two terms of (C.3) vanish separately and that the integrals of (C.3) are absolutely convergent.

We turn now to condition (3.20). From eq. (3.12) we see that E_2 is just determined by this condition. Let us calculate

$$\frac{\partial}{\partial p_\alpha} F_{\mu\nu\lambda}(p, 0)|_{p=0}.$$

From (C.3) we obtain

$$\frac{\partial}{\partial p_\alpha} F_{\mu\nu\lambda}(p, 0)|_{p=0} = \frac{ie_0^2}{(2\pi)^2} \int \mathrm{d}^2k \cdot \tag{C.5}$$

$$\cdot \operatorname{Tr} \left\{ \frac{\gamma_\mu \gamma_5 (\gamma \cdot k + m_0) \gamma_\nu (\gamma \cdot k + m_0) \gamma_\alpha (\gamma \cdot k + m_0) \gamma_\lambda (\gamma \cdot k + m_0)}{(k^2 - m_0^2)^4} + \right.$$

$$+ \frac{\gamma_\mu \gamma_5 (\gamma \cdot k + m_0) \gamma_\nu (\gamma \cdot k + m_0) \gamma_\lambda (\gamma \cdot k + m_0) \gamma_\alpha (\gamma \cdot k + m_0)}{(k^2 - m_0^2)^4} +$$

$$\left. + \frac{\gamma_\mu \gamma_5 (\gamma \cdot k + m_0) \gamma_\lambda (\gamma \cdot k + m_0) \gamma_\nu (\gamma \cdot k + m_0) \gamma_\alpha (\gamma \cdot k + m_0)}{(k^2 - m_0^2)^4} \right\} -$$

$$- i \int \frac{\mathrm{d}^2k}{(2\pi)^2} E_2^{(2)}(k) [\varepsilon_{\mu\nu} g_{\lambda\alpha} + \varepsilon_{\mu\lambda} g_{\nu\alpha}].$$

The first three terms of (C.5) give, when all nonvanishing terms are collected together,

$$\text{(C.6)}\qquad \frac{im_0^2e_0^2}{(2\pi)^2}\operatorname{Tr}\gamma_\mu\gamma_5[\gamma_\alpha\gamma_\nu\gamma_\lambda+\gamma_\alpha\gamma_\lambda\gamma_\nu+\gamma_\nu\gamma_\alpha\gamma_\lambda]\int\frac{\mathrm{d}^2k}{(k^2-m_0^2)^2}\left[2+\frac{3m_0^2}{k^2-m_0^2}\right]=0$$

if one does the integrations. From this, it follows that

$$\text{(C.7)}\qquad E_2^{(2)}(k)=0\,,$$

i.e. the lowest-order contribution vanishes. The same is true for all orders. Therefore

$$\text{(C.8)}\qquad E_2(k)\ =0\,.$$

(Apart from the proof of Sect. 4, this is true by dimensional arguments because as e carries a mass, $E_2(\xi)\sim\xi^2$, as $\xi\to0$.)

Appendix D

In the following it is explicitly shown that

$$\text{(D.1)}\qquad \Gamma_\mu^{5(0)}(p,p')\Big|_{\substack{\gamma\cdot p=\gamma\cdot p'=m_0\\ p=p'}}=\gamma_\mu\gamma_5\,.$$

From eq. (3.13) we have

$$\text{(D.2)}\qquad \Gamma_\mu^{5(0)}(p,p')=$$

$$=\gamma_\mu\gamma_5-i\int\frac{\mathrm{d}^2k}{(2\pi)^2}\gamma_\alpha\frac{1}{\gamma\cdot k-m_0}\gamma_\mu\gamma_5\frac{1}{\gamma\cdot k-\gamma\cdot p+\gamma\cdot p'-m_0}\gamma^\alpha\frac{1}{(k-p^2)-\mu_0^2}\,.$$

Doing the traces, thereby using $\gamma_\alpha\gamma_\mu\gamma^\alpha=0$, one gets

$$\Gamma_\mu^{5(0)}(p,p')=\gamma_\mu\gamma_5+\frac{2im_0}{(2\pi)^2}\int\frac{\mathrm{d}^2k[\gamma_\mu(\gamma\cdot k)-(\gamma\cdot k)\gamma_\mu+(\gamma\cdot p-\gamma\cdot p')\gamma_\mu]}{(k^2-m_0^2)[(k-p+p')^2-m_0^2][(k-p)^2-\mu_0^2]}\,.$$

Introducing Feynman parameters in the usual way and doing the k-integration gives

$$\text{(D.3)}\qquad \Gamma_\mu^{5(0)}(p,p')=\gamma_\mu\gamma_5-\frac{m_0}{2\pi}\int_0^1\mathrm{d}z\,\cdot$$

$$\cdot\int_0^1\frac{\mathrm{d}y\{\gamma_\mu[(\gamma\cdot p-\gamma\cdot p')zy+\gamma\cdot p(1-y)]-[(\gamma\cdot p-\gamma\cdot p')zy+\gamma\cdot p(1-y)]\gamma_\mu+(\gamma\cdot p-\gamma\cdot p')\gamma_\mu\}}{\{[(p-p')zy+p(1-y)]^2-(p-p')^2zy-p^2(1-y)+m_0^2y+\mu_0^2(1-y)\}^2}\,.$$

Setting $p=p'$ and putting $p=m_0$, gives immediately (D.1). It is always understood that the limit $\mu_0\to0$ has to be taken at the very end.

RIASSUNTO (*)

Usando i metodi di Wilson e Brandt si costruisce l'esatta corrente vettoriale assiale nell'elettrodinamica bidimensionale con la massa dei fermioni che non si annulla. Si dimostra che $j_\mu^{5un}(x) = \partial_\mu \varphi(x)$, dove $j_\mu^{5un}(x)$ è l'esatta corrente vettoriale assiale non rinormalizzata e $\varphi(x)$ è un campo canonico pseudoscalare. Si risolve anche il problema di costruire la lagrangiana corretta per questo modello.

(*) *Traduzione a cura della Redazione.*

Дополнительные точные результаты в двумерной спинорной электродинамике с неисчезающей фермионной массой.

Резюме (*). — Используя методы Вилсона и Брандта, конструируется точный аксиальный векторный ток в двумерной электродинамике с неисчезающей фермионной массой. Показывается, что $j_\mu^{5un}(x) = \partial_\mu \varphi(x)$, где $j_\mu^{5un}(x)$ представляет неперенормированный аксиальный векторный ток, а $\varphi(x)$ есть псевдоскалярное каноническое поле. Также решается проблема конструирования правильного Лагранжиана для этой модели.

(*) *Переведено редакцией.*